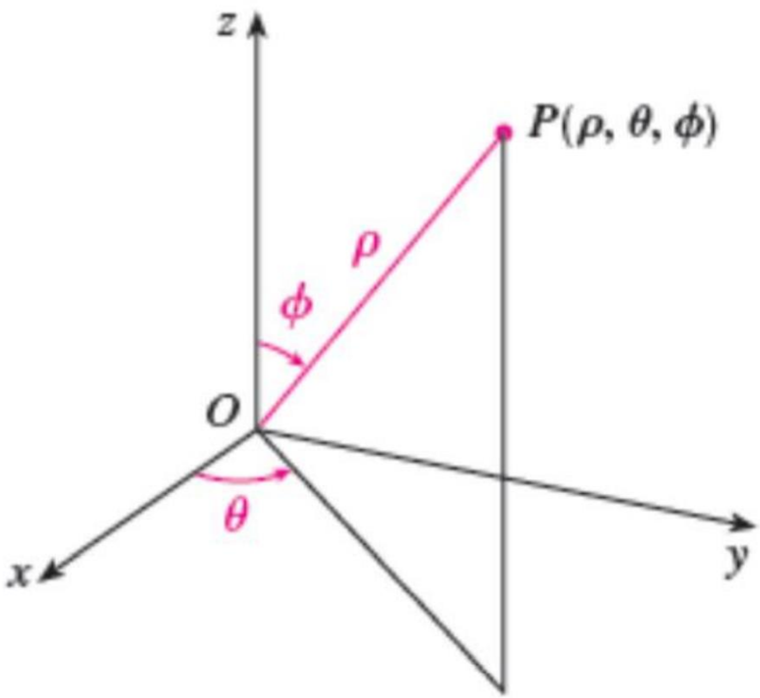
Section 15.8:
Triple Integrals In Spherical Coordinates

What We’ll Learn In Section 15.8

1. What are spherical coordinates?
2. Triple integrals in spherical coordinates

1. What are spherical coordinates?

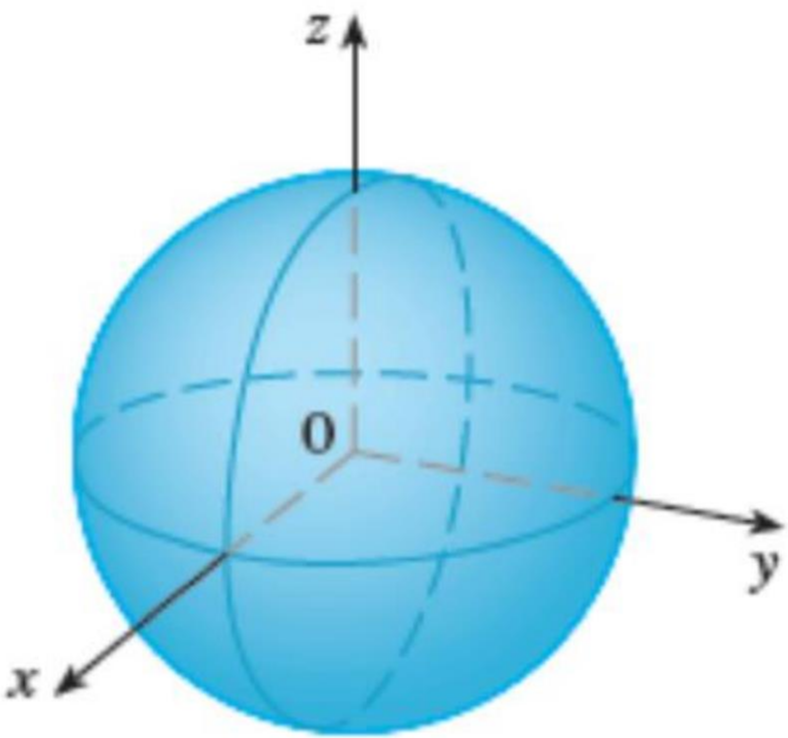
The spherical coordinates of a point



$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

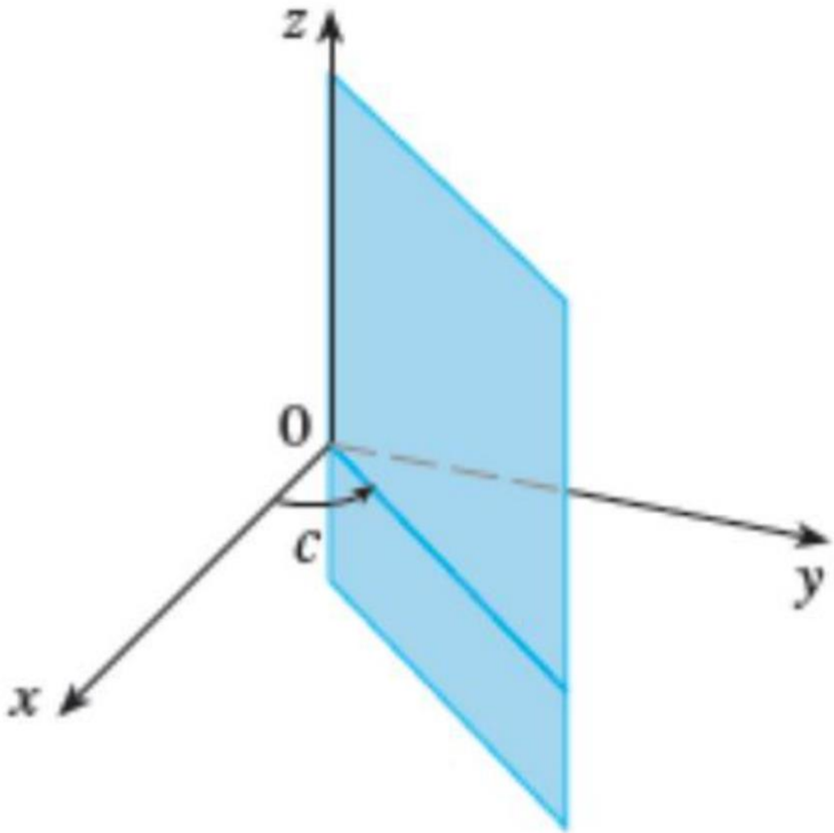
1. What are spherical coordinates?

$\rho = c$, a sphere



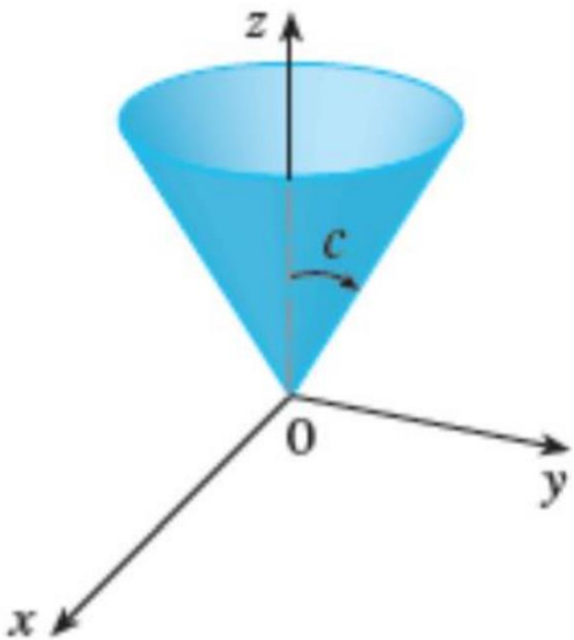
1. What are spherical coordinates?

$\theta = c$, a half-plane

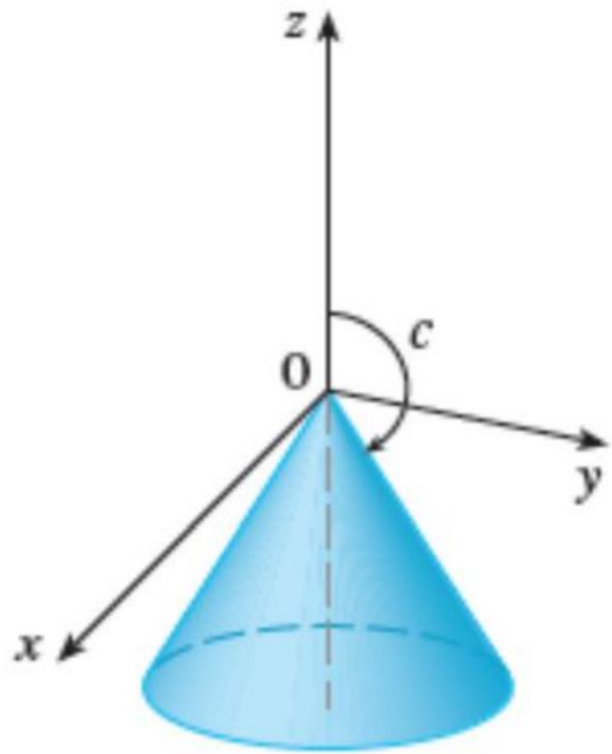


1. What are spherical coordinates?

$\phi = c$, a half-cone



$0 < c < \pi/2$

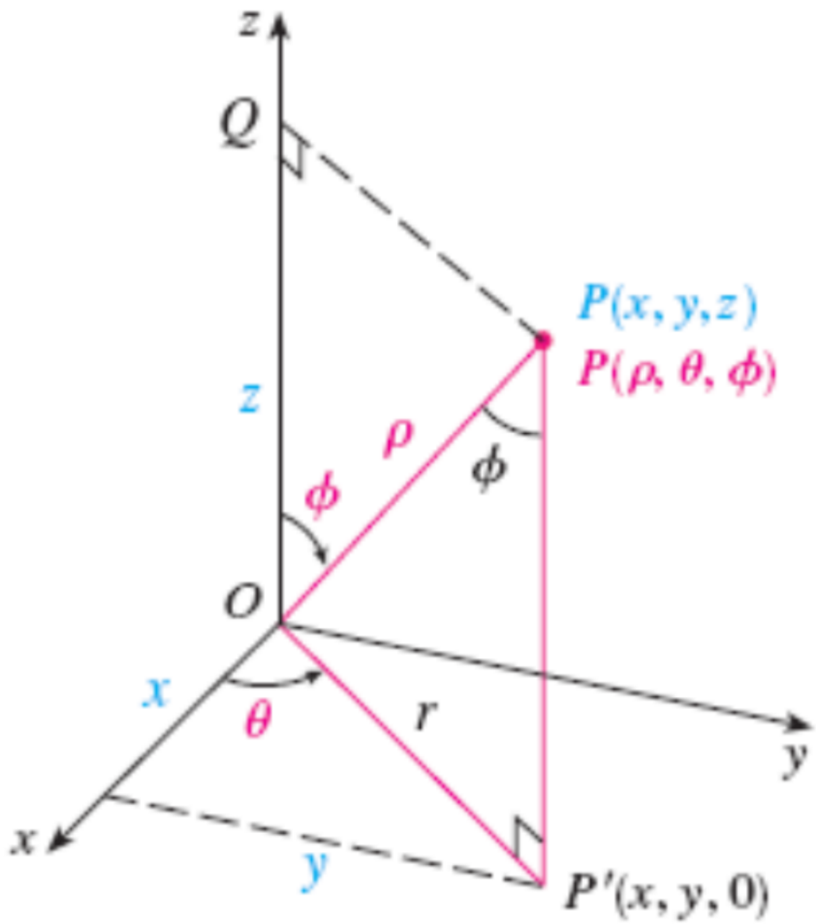


$\pi/2 < c < \pi$

1. What are spherical coordinates?

$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$



$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

1. What are spherical coordinates?

Ex 1: The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

1. What are spherical coordinates?

Warning There is not universal agreement on the notation for spherical coordinates. Most books on physics reverse the meanings of θ and ϕ and use r in place of ρ.

1. What are spherical coordinates?

Ex 2: The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

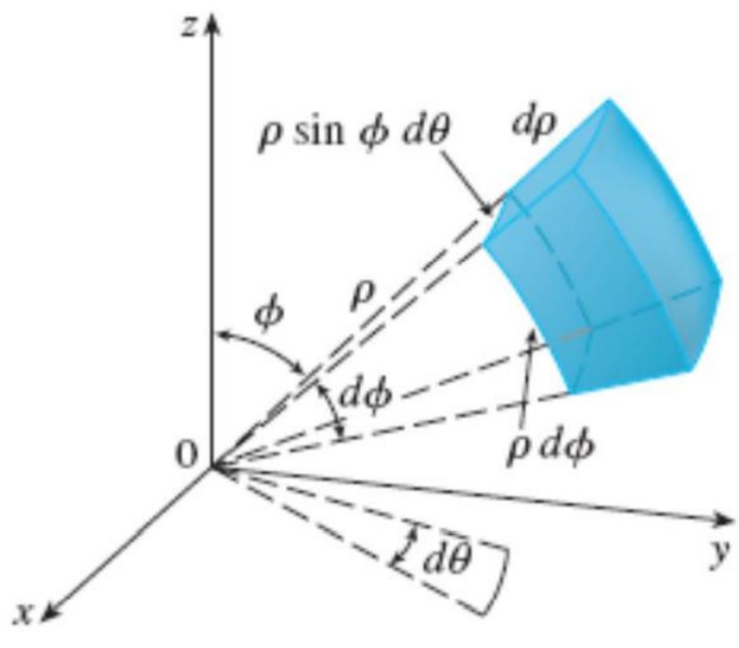
2. Triple integrals in spherical coordinates

To integrate in spherical coordinates…

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Volume element in spherical coordinates:

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$



2. Triple integrals in spherical coordinates

To integrate in spherical coordinates…

$$\iiint\limits_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid \ a \leqslant \rho \leqslant b, \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d\}$$

2. Triple integrals in spherical coordinates

To integrate in spherical coordinates…

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d, g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

2. Triple integrals in spherical coordinates

Ex 3: Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball $B = \{\, (x, y, z) \mid x^2 + y^2 + z^2 \leq 1 \,\}$.

In rectangular coordinates, this integral becomes…

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

2. Triple integrals in spherical coordinates

Ex 4: Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.